

Genus one 1-bridge knots and Dunwoody manifolds*

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Abstract

In this paper we show that all 3-manifolds of a family introduced by M. J. Dunwoody are cyclic coverings of lens spaces (eventually \mathbf{S}^3), branched over genus one 1-bridge knots. As a consequence, we give a positive answer to the Dunwoody conjecture that all the elements of a wide subclass are cyclic coverings of \mathbf{S}^3 branched over a knot. Moreover, we show that all branched cyclic coverings of a 2-bridge knot belong to this subclass; this implies that the fundamental group of each branched cyclic covering of a 2-bridge knot admits a geometric cyclic presentation.

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1 Introduction and preliminaries

The problem of determining if a balanced presentation of a group is geometric (i.e. induced by a Heegaard diagram of a closed orientable 3-manifold) is quite important within geometric topology and has been deeply investigated

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by many authors (see [9], [22], [25], [26], [27], [28], [33]); further, the connections between branched cyclic coverings of links and cyclic presentations of groups induced by suitable Heegaard diagrams have been recently pointed out in several papers (see [1], [3], [4], [6], [11], [12], [16], [18], [17], [20], [34]). In order to investigate these connections, M.J. Dunwoody introduces in [6] a class of planar, 3-regular graphs endowed with a cyclic symmetry. Each graph is defined by a 6-tuple of integers; if this 6-tuple satisfies suitable conditions (admissible 6-tuple), the graph uniquely defines a Heegaard diagram such that the presentation of the fundamental group of the represented manifold is cyclic. This construction gives rise to a wide class of closed orientable 3-manifolds (Dunwoody manifolds), depending on 6-tuples of integers and admitting geometric cyclic presentations for their fundamental groups. Our main result is that each Dunwoody manifold is a cyclic covering of a lens space (eventually the 3-sphere), branched over a genus one 1-bridge knot. As a direct consequence, the Dunwoody manifolds belonging to a wide subclass are proved to be cyclic coverings of S^3 , branched over suitable knots, thus giving a positive answer to a conjecture of Dunwoody [6]. Moreover, we show that all branched cyclic coverings of knots with classical (i.e. genus zero) bridge number two belong to this subclass; as a corollary, the fundamental group of each branched cyclic covering of a 2-bridge knot admits a geometric cyclic presentation.

For the theory of Heegaard splittings of 3-manifolds, and in particular for Singer moves on Heegaard diagrams realizing the homeomorphism of the represented manifolds, we refer to [13] and [31]. For the theory of cyclically presented groups, we refer to [15].

We recall that a finite balanced presentation of a group $\langle x_1, \dots, x_n | r_1, \dots, r_n \rangle$ is said to be a *cyclic presentation* if there exists a word w in the free group F_n generated by x_1, \dots, x_n such that the relators of the presentation are $r_k = \theta_n^{k-1}(w)$, $k = 1, \dots, n$, where $\theta_n : F_n \rightarrow F_n$ denotes the automorphism defined by $\theta_n(x_i) = x_{i+1} \pmod{n}$, $i = 1, \dots, n$. Let us denote this cyclic presentation (and the related group) by the symbol $G_n(w)$, so that:

$$G_n(w) = \langle x_1, x_2, \dots, x_n | w, \theta_n(w), \dots, \theta_n^{n-1}(w) \rangle.$$

A group is said to be cyclically presented if it admits a cyclic presentation. We recall that the *exponent-sum* of a word $w \in F_n$ is the integer ε_w given by the sum of the exponents of its letters; in other terms, $\varepsilon_w = v(w)$ where $v : F_n \rightarrow \mathbf{Z}$ is the homomorphism defined by $v(x_i) = 1$ for each $1 \leq i \leq n$.

Following [10], we recall the definition of genus g bridge number of a link, which is a generalization of the classical concept of bridge number for links in \mathbf{S}^3 (see [5]).

A set of mutually disjoint arcs $\{t_1, \dots, t_n\}$ properly embedded in a handlebody U is trivial if there is a set of mutually disjoint discs $D = \{D_1, \dots, D_n\}$ such that $t_i \cap D_i = t_i \cap \partial D_i = t_i$, $t_i \cap D_j = \emptyset$ and $\partial D_i - t_i \subset \partial U$ for $1 \leq i, j \leq n$ and $i \neq j$. Let U_1 and U_2 be the two handlebodies of a Heegaard splitting of the closed orientable 3-manifold M and let T be their common surface: a link L in M is in n -bridge position with respect to T if L intersects T transversally and if the set of arcs $L \cap U_i$ has n components and is trivial both in U_1 and in U_2 . A link in 1-bridge position is obviously a knot.

The genus g bridge number of a link L in M , $b_g(L)$, is the smallest integer n for which L is in n -bridge position with respect to some genus g Heegaard surface in M . If the genus g bridge number of a link L is b , we say that L is a genus g b -bridge link or simply a (g, b) -link. Of course, the genus g bridge number of a link in a manifold of Heegaard genus g' is defined only for $g \geq g'$ and the genus 0 bridge number of a link in \mathbf{S}^3 is the classical bridge number. Moreover, a $(g, 1)$ -link is a knot, for each $g \geq 0$.

In what follows, we shall deal with $(1, 1)$ -knots, i.e. knots in \mathbf{S}^3 or in lens spaces. This class of knots is very important in the light of some results and conjectures involving Dehn surgery on knots (see [2], [7], [8], [35], [36], [37]). Notice that the class of $(1, 1)$ -knots in \mathbf{S}^3 contains all torus knots (trivially) and all 2-bridge knots (i.e. $(0, 2)$ -knots) [23].

2 Dunwoody manifolds

Let us sketch now the construction of Dunwoody manifolds given in [6]. Let a, b, c, n be integers such that $n > 0$, $a, b, c \geq 0$ and $a + b + c > 0$. Let $\Gamma = \Gamma(a, b, c, n)$ be the planar regular trivalent graph drawn in Figure 1.

It contains n upper cycles C'_1, \dots, C'_n and n lower cycles C''_1, \dots, C''_n , each having $d = 2a + b + c$ vertices. For each $i = 1, \dots, n$, the cycle C'_i (resp. C''_i) is connected to the cycle C'_{i+1} (resp. C''_{i+1}) by a parallel arcs, to the cycle C''_i by c parallel arcs and to the cycle C''_{i+1} by b parallel arcs (assume $n + 1 = 1$). We set $\mathcal{C}' = \{C'_1, \dots, C'_n\}$ and $\mathcal{C}'' = \{C''_1, \dots, C''_n\}$. Moreover, denote by A' (resp. A'') the set of the arcs of Γ belonging to a cycle of \mathcal{C}' (resp. \mathcal{C}'') and by

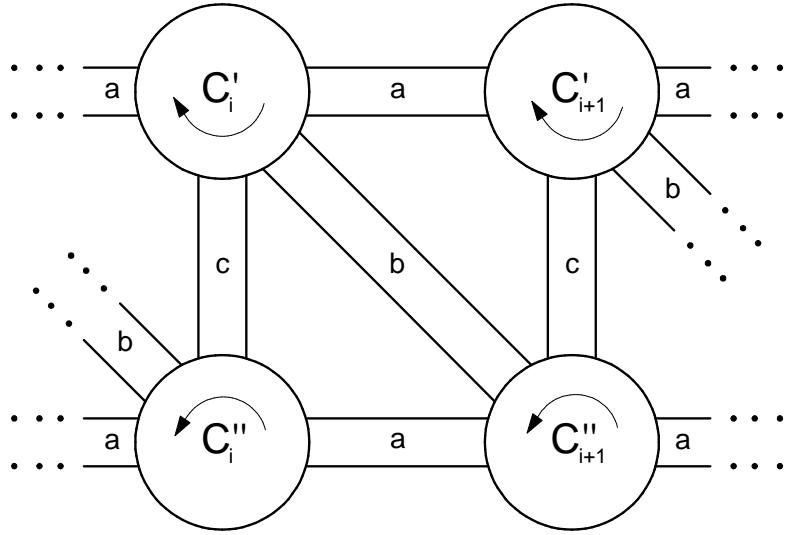


Figure 1: The graph Γ .

A the set of the other arcs of the graph. The one-point compactification of the plane leads to a 2-cell embedding of the graph Γ in S^2 ; it is evident that the graph is invariant with respect to a rotation ρ_n of the sphere by $2\pi/n$ radians along a suitable axis intersecting S^2 in two points not belonging to the graph. Obviously, ρ_n sends C'_i to C'_{i+1} and C''_i to C''_{i+1} ($\bmod n$), for each $i = 1, \dots, n$.

By cutting the sphere along all C'_i and C''_i and by removing the interior of the corresponding discs, we obtain a sphere with $2n$ holes. Let now r and s be two new integers; give a clockwise (resp. counterclockwise) orientation to the cycles of \mathcal{C}' (resp. of \mathcal{C}'') and label their vertices from 1 to d , in accordance with these orientations (see Figure 2) so that:

- the vertex 1 of each C'_i is the endpoint of the first arc of A connecting C'_i with C'_{i+1} ;
- the vertex $1 - r \pmod d$ of each C''_i is the endpoint of the first arc of A connecting C''_i with C''_{i+1} .

Then glue the cycle C'_i with the cycle $C''_{i-s} \pmod n$ so that equally labelled vertices are identified together.

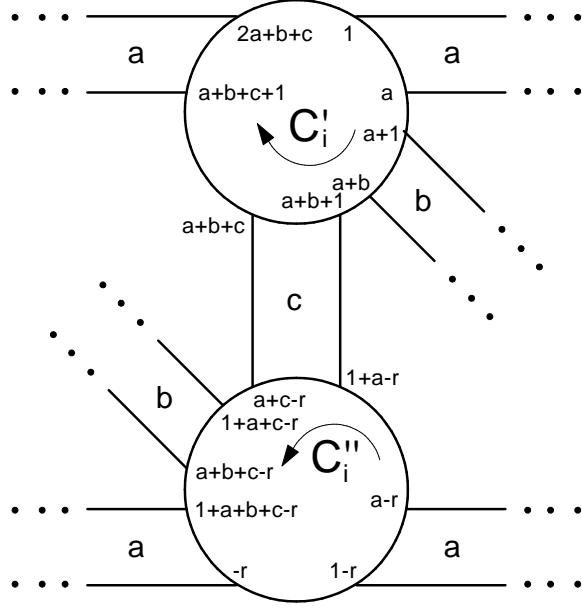


Figure 2:

It is evident by construction that the integers r and s can be taken mod d and mod n respectively. Denote by \mathcal{S} the set of all the 6-tuples $(a, b, c, n, r, s) \in \mathbf{Z}^6$ such that $n > 0$, $a, b, c \geq 0$ and $a + b + c > 0$.

The described gluing gives rise to an orientable surface T_n of genus n and the nd arcs belonging to A are pairwise connected through their endpoints, realizing m cycles D_1, \dots, D_m on T_n . It is straightforward that the cut of T_n along the n cycles $C_i = C'_i = C''_{i-s}$ does not disconnect the surface. Set $\mathcal{C} = \{C_1, \dots, C_n\}$ and $\mathcal{D} = \{D_1, \dots, D_m\}$.

If $m = n$ and if the cut along the cycles of \mathcal{D} does not disconnect T_n , then the two systems of meridian curves \mathcal{C} and \mathcal{D} in T_n represent a genus n Heegaard diagram of a closed orientable 3-manifold, which is completely determined by the 6-tuple. Each manifold arising in this way is called a *Dunwoody manifold*.

Thus, we define to be *admissible* the 6-tuples (a, b, c, n, r, s) of \mathcal{S} satisfying the following conditions:

- (1) the set \mathcal{D} contains exactly n cycles;
- (2) the surface T_n is not disconnected by the cut along the cycles of \mathcal{D} .

The “open” Heegaard diagram Γ and the Dunwoody manifold associated to the admissible 6-tuple σ will be denoted by $H(\sigma)$ and $M(\sigma)$ respectively.

Remark 1. It is easy to see that not all the 6-tuples in \mathcal{S} are admissible. For example, the 6-tuples $(a, 0, a, 1, a, 0)$, with $a \geq 1$, give rise to exactly a cycles in \mathcal{D} ; thus, they are not admissible if $a > 1$. The 6-tuples $(1, 0, c, 1, 2, 0)$ are not admissible if c is even, since, in this case, we obtain exactly one cycle D_1 , but the cut along it disconnects the torus T_1 .

Consider now a 6-tuple $\sigma \in \mathcal{S}$. The graph Γ becomes, via the gluing quotient map, a regular 4-valent graph denoted by Γ' embedded in T_n . Its vertices are the intersection points of the spaces $\Omega = \cup_{i=1}^n C_i$ and $\Lambda = \cup_{j=1}^m D_j$; hence they inherit the labelling of the corresponding glued vertices of Γ . Since the gluing of the cycles of \mathcal{C}' and \mathcal{C}'' is invariant with respect to the rotation ρ_n , the group $\mathcal{G}_n = \langle \rho_n \rangle$ naturally induces a cyclic action of order n on T_n such that the quotient $T_1 = T_n / \mathcal{G}_n$ is homeomorphic to a torus. The labelling of the vertices of Γ' is invariant under the rotation ρ_n and $\rho_n(C_i) = C_{i+1} \pmod{n}$. We are going to show that, if the 6-tuple is admissible, this last property also holds for the cycles of \mathcal{D} .

Lemma 1 a) Let $\sigma = (a, b, c, n, r, s)$ be an admissible 6-tuple. Then ρ_n induces a cyclic permutation on the curves of \mathcal{D} . Thus, if D is a cycle of \mathcal{D} , then $\mathcal{D} = \{\rho_n^{k-1}(D) | k = 1, \dots, n\}$.

b) If (a, b, c, n, r, s) is admissible, then also $(a, b, c, 1, r, 0)$ is admissible and the Heegaard diagram $H(a, b, c, 1, r, 0)$ is the quotient of the Heegaard diagram $H(a, b, c, n, r, s)$ respect to \mathcal{G}_n .

Proof. a) First of all, note that $\rho_n(\Lambda) = \Lambda$; thus the group \mathcal{G}_n also acts on the spaces $T_n - \Lambda$ and Λ (and hence on the set \mathcal{D}). If the 6-tuple σ is admissible, then $T_n - \Lambda$ is connected, and hence the quotient $(T_n - \Lambda) / \mathcal{G}_n = T_n / \mathcal{G}_n - \Lambda / \mathcal{G}_n$ must be connected too. This implies that Λ / \mathcal{G}_n has a unique connected component. Since Λ has exactly n connected components, the cyclic group \mathcal{G}_n of order n defines a simply transitive cyclic action on the cycles of \mathcal{D} .

b) Let $C, D \subset T_1$ the two curves $C = \Omega / \mathcal{G}_n$ and $D = \Lambda / \mathcal{G}_n$. Then, the two systems of curves $\mathcal{C} = \{C\}$ and $\mathcal{D} = \{D\}$ on T_1 define a Heegaard diagram of genus one. The graph Γ_1 corresponding to $\sigma_1 = (a, b, c, 1, r, 0)$ is the quotient of the graph Γ_n corresponding to $\sigma = (a, b, c, n, r, s)$, respect to \mathcal{G}_n . Moreover, the gluings on Γ_n are invariant respect to ρ_n . Therefore,

the gluings on Γ_1 give rise to the Heegaard diagram above. This show that the 6-tuple σ_1 is admissible and obviously $H(a, b, c, 1, r, 0)$ is the quotient of $H(a, b, c, n, r, s)$ respect to \mathcal{G}_n . ■

Remark 2. More generally, given two positive integer n and n' such that n' divides n , if (a, b, c, r, n, s) is admissible, then (a, b, c, r, n', b) is admissible too. Moreover, the Heegaard diagram $H(a, b, c, r, n', b)$ is the quotient of $H(a, b, c, r, n, b)$ respect to the action of a cyclic group of order n/n' .

It is easy to see that, for admissible 6-tuples, each cycle in \mathcal{D} contains d vertices with different labels and is composed by exactly d arcs of Γ (in fact, $2a$ horizontal arcs, b oblique arcs and c vertical arcs).

An important consequence of point a) of Lemma 1 is that, if σ is an admissible 6-tuple, the presentation of the fundamental group of $M(\sigma)$ induced by the Heegaard diagram $H(\sigma)$ is cyclic.

To see this, let v be the vertex belonging to the cycle C_1 and labelled by $a + b + 1$; denote by D_1 the curve of \mathcal{D} containing v and by v' the vertex of C'_1 corresponding to v . Orient the arc $e' \in A$ of the graph Γ containing v' so that v' is its first endpoint and orient the curve D_1 in accordance with the orientation of this arc. Now, set $D_k = \rho_n^{k-1}(D_1)$, for each $k = 1, \dots, n$; the orientation on D_1 induces, via ρ_n , an orientation also on these curves. Moreover, these orientation on the cycles of \mathcal{D} induce an orientation on the arcs of the graph Γ belonging to A . By orienting the arcs of C' and C'' in accordance with the fixed orientations of the cycles C'_i and C''_i , the graph Γ becomes an oriented graph, whose orientation is invariant under the action of the group \mathcal{G}_n . Let us define to be *canonical* this orientation of Γ .

Let now $w \in F_n$ be the word obtained by reading the oriented arcs $e_1 = e', e_2, \dots, e_d$ of Γ corresponding to the oriented cycle D_1 , starting from the vertex v' . The letters of w are in one-to-one correspondence with the oriented arcs e_h ; more precisely, the letter of w corresponding to e_h is x_i if e_h comes out from the cycle C'_i and is x_i^{-1} if e_h comes out from the cycle C''_{i-s} . Note that the word $\theta_n^{k-1}(w)$ in the cyclic presentation $G_n(w)$ is obtained by reading the cycle D_k along the given orientation, for $1 \leq k \leq n$ (roughly speaking, the automorphism θ_n is “geometrically” realized by ρ_n).

This proves that each admissible 6-tuple σ uniquely defines, via the associated Heegaard diagram $H(\sigma)$, a word $w = w(\sigma)$ and a cyclic presentation $G_n(w)$ for the fundamental group of the Dunwoody manifold $M(\sigma)$. Note that the sequence of the exponents in the word $w(\sigma)$, and hence its exponent-sum $\varepsilon_{w(\sigma)}$, only depends on the integers a, b, c, r .

Let us consider now the Dunwoody manifolds $M(a, b, c, n, r, s)$ with $n = 1$ (and hence $s = 0$), which arises from a genus one Heegaard diagram.

Proposition 2 *Let $(a, b, c, 1, r, 0)$ be an admissible 6-tuple and let $w = w(a, b, c, 1, r, 0)$ be the associated word. Then the Dunwoody manifold $M(a, b, c, 1, r, 0)$ is homeomorphic to:*

- i) \mathbf{S}^3 , if $\varepsilon_w = \pm 1$;
- ii) $\mathbf{S}^1 \times \mathbf{S}^2$, if $\varepsilon_w = 0$;
- iii) a lens space $L(\alpha, \beta)$ with $\alpha = |\varepsilon_w|$, if $|\varepsilon_w| > 1$.

Proof. From $n = 1$ we obtain $w \in F_1 \cong \mathbf{Z} \cong \langle x | \emptyset \rangle$. Thus, $\pi_1(M) \cong G_1(w) \cong \langle x | x^{\varepsilon_w} \rangle \cong \mathbf{Z}_{|\varepsilon_w|}$. ■

Example 1. The Dunwoody manifolds $M(0, 0, 1, 1, 0, 0)$, $M(1, 0, 0, 1, 1, 0)$ and $M(0, 0, c, 1, r, 0)$, with c, r coprime, are homeomorphic to \mathbf{S}^3 , $\mathbf{S}^1 \times \mathbf{S}^2$ and to the lens space $L(c, r)$, respectively. Moreover, all lens spaces also arise with $a \neq 0$; in fact, for each $a > 0$, $M(a, 0, c, 1, a, 0)$ is homeomorphic with the lens space $L(c, a)$, if a and c are coprime, since it is easy to see that $H(a, 0, c, 1, a, 0)$ can be transformed into the canonical genus one Heegaard diagram of $L(c, a)$ by Singer moves of type IB.

Let us see now how the admissibility conditions for the 6-tuples of \mathcal{S} can be given in terms of labelling of the vertices of Γ' , belonging to the curve $D_1 \in \mathcal{D}$. With this aim, consider the following properties for a 6-tuple $\sigma \in \mathcal{S}$:

- (i') the set of the labels of the vertices belonging to the cycle D_1 is the set of all integers from 1 to d ;
- (ii') the vertices of the cycle D_1 have different labels.

It is easy to see that, if a 6-tuple $\sigma \in \mathcal{S}$ is admissible, then it satisfies (i') and (ii'). On the other side, if a 6-tuple $\sigma \in \mathcal{S}$ satisfies (i') and (ii'), then the curves $\rho_n^{k-1}(D_1) \in \mathcal{D}$, with $k = 1, \dots, n$, which are all different from each other, are precisely the curves of \mathcal{D} . Thus, \mathcal{D} has exactly n curves and they are cyclically permuted by ρ_n . However, this does not imply that σ is admissible; for example, the 6-tuple $(1, 0, 2, 1, 2, 0)$ satisfies (i') and (ii'), but it is not admissible (see Remark 1). Note that, for $n = 1$, property (ii') always holds, while condition (i') holds if and only if \mathcal{D} has a unique cycle.

If a 6-tuple satisfies property (i'), then \mathcal{G}_n acts transitively (not necessarily simply) on \mathcal{D} , and hence it is possible to induce an orientation (which is still

said to be canonical) on the cycles of \mathcal{D} and on the graph Γ , by extending, via ρ_n , the orientation of D_1 to the other cycles of \mathcal{D} .

Property (i') implies that the cycles of \mathcal{D} naturally induce a cyclic permutation on the set $\mathcal{N} = \{1, \dots, d\}$ of the vertex labels. In fact, by walking along these canonically oriented cycles, starting from an arbitrary vertex \bar{v} labelled j , one sequentially meets d vertices (whose labels are different from each other), and then a new vertex \bar{v}' labelled j which can be different from \bar{v} . The sequence of the labellings of these d consecutive vertices defines the cyclic permutation on \mathcal{N} . Further, each cycle of \mathcal{D} precisely contains $d' = ld$ arcs, with $l \geq 1$, and $l = 1$ if and only if the 6-tuple satisfies (ii') too. Moreover, property (i') is independent from the integers n and s ; hence, given two 6-tuples $\sigma = (a, b, c, n, r, s)$ and $\sigma' = (a, b, c, n', r, s)$, then σ satisfies (i') if and only if σ' satisfies (i').

Let now σ be a 6-tuple satisfying (i') and suppose that Γ is canonically oriented. An arc of Γ belonging to A is said to be of type I if it is oriented from a cycle of \mathcal{C}' to a cycle of \mathcal{C}'' , of type II if it is oriented from a cycle of \mathcal{C}'' to a cycle of \mathcal{C}' and of type III otherwise (it joins cycles of \mathcal{C}' or cycles of \mathcal{C}''). Moreover, the arc is said to be of type I' if it is oriented from a cycle C'_i (resp. C''_i) to a cycle C'_{i+1} (resp. C''_{i+1}), of type II' if it is oriented from a cycle C'_{i+1} (resp. C''_{i+1}) to a cycle C'_i (resp. C''_i) and of type III' otherwise (it joins C'_i with C''_i). Let Δ be the set of the first d arcs of D_1 , following the canonical orientation, starting from the arc coming out from the vertex v' of C'_1 labelled $a+b+1$. Obviously, the set Δ contains all the arcs of D_1 if and only if the 6-tuple σ also satisfies (ii').

Now, denote by p'_σ (resp. p''_σ) the number of the arcs of type I (resp. of type II) of Δ and set $p_\sigma = p'_\sigma - p''_\sigma$. Similarly, denote by q'_σ (resp. q''_σ) the number of the arcs of type I' (resp. of type II') of Δ and set $q_\sigma = q'_\sigma - q''_\sigma$. Note that p_σ has the same parity of $b+c$ and q_σ has the same parity of $2a+b$ and hence of b . It is evident that p_σ and q_σ only depend on the integers a, b, c, r .

The integers p_σ and q_σ give an useful tool for verifying condition (ii'). In fact, suppose to walk along the canonically oriented cycle D_j of \mathcal{D} , starting from a vertex \bar{v} and let C_i be the cycle of \mathcal{C} containing \bar{v} . If \bar{v}' is the first vertex with the same label of \bar{v} and if $C_{i'}$ is the cycle of \mathcal{C} containing \bar{v}' , we have $i' = i + q_\sigma + sp_\sigma$. Thus, the cycle D_j contains d arcs if and only if $q_\sigma + sp_\sigma \equiv 0 \pmod{n}$. This proves that the 6-tuple satisfies (ii'). Thus, (i') and (ii') are respectively, in a different language, conditions (i) and (ii) of Theorem 2 of [6], which gives a necessary and sufficient condition for a

6-tuple to be admissible when d is odd. In fact, we have the following result:

Lemma 3 ([6], Theorem 2) *Let $\sigma = (a, b, c, n, r, s)$ be a 6-tuple with $d = 2a + b + c$ odd. Then σ is admissible if and only if it satisfies (i') and (ii').*

Remark 3. This result does not hold when d is even. In fact, the 6-tuples $(1, 0, c, 1, 2, 0)$, with c even, satisfy (i') and (ii'), but they are not admissible, as pointed out in Remark 1.

An immediate consequence of Lemma 3 is the following result:

Corollary 4 *Let $\sigma = (a, b, c, n, r, s)$ be a 6-tuple with $d = 2a + b + c$ odd and $n = 1$. Then σ is admissible if and only if \mathcal{D} has a unique cycle.*

Proof. If σ is admissible, then it is straightforward that \mathcal{D} has a unique cycle. Vice versa, if \mathcal{D} has a unique cycle, then (i') holds. Since $n = 1$ implies (ii'), the result is a direct consequence of the above lemma. ■

The parameter p_σ associated to an admissible 6-tuple σ is strictly related to the word $w(\sigma)$ associated to σ . In fact, we have:

Lemma 5 *Let $\sigma = (a, b, c, n, r, s)$ be an admissible 6-tuple, $w = w(\sigma)$ the associated word and ε_w its exponent-sum. Then*

$$p_\sigma = \varepsilon_w.$$

Proof. Since σ is admissible, the arcs of Δ are precisely the arcs of D_1 . Let e_1, e_2, \dots, e_d be the sequence of these arcs, following the canonical orientation on D_1 , and let $w = \prod_{h=1}^d x_{i_h}^{u_h}$, with $u_h \in \{+1, -1\}$. We have: $\varepsilon_w = \sum_{h=1}^d u_h = 1/2 \sum_{h=1}^d (u_h + u_{h+1})$, where $d+1 = 1$. Since $u_h + u_{h+1} = +2$ if e_h is of type I, $u_h + u_{h+1} = -2$ if e_h is of type II and $u_h + u_{h+1} = 0$ if e_h is of type III, the result immediately follows. ■

In [6] Dunwoody investigates a wide subclass of manifolds $M(\sigma)$ such that $p_\sigma = \pm 1$ and he conjectures that all the elements of this subclass are cyclic coverings of \mathbf{S}^3 branched over knots. In the next chapter this conjecture will be proved as a corollary of a more general theorem.

3 Main results

The following theorem is the main result of this paper and shows how the cyclic action on the Heegaard diagrams naturally extends to a cyclic action on the associated Dunwoody manifolds, which turn out to be cyclic coverings of \mathbf{S}^3 or of lens spaces, branched over suitable knots.

Theorem 6 *Let $\sigma = (a, b, c, n, r, s)$ be an admissible 6-tuple, with $n > 1$. Then the Dunwoody manifold $M = M(a, b, c, n, r, s)$ is the n -fold cyclic covering of the manifold $M' = M(a, b, c, 1, r, 0)$, branched over a genus one 1-bridge knot $K = K(a, b, c, r)$ only depending on the integers a, b, c, r . Further, M' is homeomorphic to:*

- i) \mathbf{S}^3 , if $p_\sigma = \pm 1$,
- ii) $\mathbf{S}^1 \times \mathbf{S}^2$, if $p_\sigma = 0$,
- iii) a lens space $L(\alpha, \beta)$ with $\alpha = |p_\sigma|$, if $|p_\sigma| > 1$.

Proof. Since the two systems of curves $\mathcal{C} = \{C_1, \dots, C_n\}$ and $\mathcal{D} = \{D_1, \dots, D_n\}$ on T_n define a Heegaard diagram of M , there exist two handlebodies U_n and U'_n of genus n , with $\partial U_n = \partial U'_n = T_n$, such that $M = U_n \cup U'_n$. Let now \mathcal{G}_n be the cyclic group of order n generated by the homeomorphism ρ_n on T_n . The action of \mathcal{G}_n on T_n extends to both the handlebodies U_n and U'_n (see [29]), and hence to the 3-manifold M . Let B_1 (resp. B'_1) be a disc properly embedded in U_n (resp. in U'_n) such that $\partial B_1 = C_1$ (resp. $\partial B'_1 = D_1$). Since $\rho_n(C_i) = C_{i+1}$ and $\rho_n(D_i) = D_{i+1}$ (mod n), the discs $B_k = \rho_n^{k-1}(B_1)$ (resp. $B'_k = \rho_n^{k-1}(B'_1)$), for $k = 1, \dots, n$, form a system of meridian discs for the handlebody U_n (resp. U'_n). By arguments contained in [38], the quotients $U_1 = U_n/\mathcal{G}_n$ and $U'_1 = U'_n/\mathcal{G}_n$ are both handlebody orbifolds topologically homeomorphic to a genus one handlebody with one arc trivially embedded as its singular set with a cyclic isotropy group of order n . The intersection of these orbifolds is a 2-orbifold with two singular points of order n , which is topologically the torus $T_1 = T_n/\mathcal{G}_n$; the curve C (resp. D), which is the image via the quotient map of the curves C_i (resp. of the curves D_i), is non-homotopically trivial in T_1 . These curves, each of which is a fundamental system of curves in T_1 , define a Heegaard diagram of M' (induced by $H(a, b, c, 1, r, 0)$). The union of the orbifolds U_1 and U'_1 is a 3-orbifold topologically homeomorphic to M' , having a genus one 1-bridge knot $K \subset M'$ as singular set of order n . Thus, M' is homeomorphic to M/\mathcal{G}_n and hence M is the n -fold cyclic covering of M' , branched over K .

Since the handlebody orbifolds and their gluing only depend on a, b, c, r , the same holds for the branching set K . The homeomorphism type of M' follows from Proposition 2 and Lemma 5. ■

Remark 4. More generally, given two positive integers n and n' such that n' divides n , if (a, b, c, r, n, s) is admissible, then the Dunwoody manifold $M(a, b, c, n, r, s)$ is the n/n' -fold cyclic covering of the manifold $M' = M(a, b, c, n', r, s)$, branched over an $(n', 1)$ -knot in M' .

Example 2. The Dunwoody manifolds $M(0, 0, 1, n, 0, 0)$, $M(1, 0, 0, n, 1, 0)$ and $M(0, 0, c, n, r, 0)$, with c, r coprime, are n -fold cyclic coverings of the manifolds \mathbf{S}^3 , $\mathbf{S}^1 \times \mathbf{S}^2$ and $L(c, r)$ respectively, branched over a trivial knot. In fact, these Dunwoody manifolds are the connected sum of n copies of \mathbf{S}^3 , $\mathbf{S}^1 \times \mathbf{S}^2$ and $L(c, r)$ respectively.

Let us consider now the class of the Dunwoody manifolds $M_n = M(a, b, c, n, r, s)$ with $p = \pm 1$ (and hence d odd) and $s = -pq$. Many examples of these manifolds appear in Table 1 of [6], where it was conjectured that they are n -fold cyclic coverings of \mathbf{S}^3 , branched over suitable knots. The following corollary of Theorem 6 proves this conjecture.

Corollary 7 *Let $\sigma_1 = (a, b, c, 1, r, 0)$ be an admissible 6-tuple with $p_{\sigma_1} = \pm 1$ and $s = -p_{\sigma_1}q_{\sigma_1}$. Then the 6-tuple $\sigma_n = (a, b, c, n, r, s)$ is admissible for each $n > 1$ and the Dunwoody manifold $M_n = M(a, b, c, n, r, s)$ is a n -fold cyclic coverings of \mathbf{S}^3 , branched over a genus one 1-bridge knot $K \subset \mathbf{S}^3$, which is independent on n .*

Proof. Obviously $(a, b, c, 1, r, s) = \sigma_1$. Since σ_1 is admissible, it satisfies (i'). This proves that σ_n satisfies (i'), for each $n > 1$. Since $s = -p_{\sigma_1}q_{\sigma_1} = -p_{\sigma_n}q_{\sigma_n}$ and $p_{\sigma_n} = p_{\sigma_1} = \pm 1$, we obtain $q_{\sigma_n} + sp_{\sigma_n} = 0$, for each $n > 1$, which implies condition (ii) of Theorem 2 of [6], or equivalently (ii'). Moreover, d is odd, since $[d]_2 = [2a + b + c]_2 = [b + c]_2 = [p_{\sigma_n}]_2 = [p_{\sigma_1}]_2 = 1$. Thus, Lemma 3 proves that σ_n is admissible. The final result is then a direct consequence of Theorem 6. ■

We point out that the above result has been independently obtained by H. J. Song and S. H. Kim in [32].

An interesting problem which naturally arises is that of characterizing the set \mathcal{K} of branching knots in \mathbf{S}^3 involved in Corollary 7. The next theorem shows that it contains all 2-bridge knots. We recall that a 2-bridge knot is determined by two coprime integers α and β , with $\alpha > 0$ odd. The

classification of 2-bridge knots and links has been obtained by Schubert in [30]. Since the 2-bridge knot of type (α, β) is equivalent to the 2-bridge knot of type $(\alpha, \alpha - \beta)$, then β can be assumed to be even.

Theorem 8 *The 6-tuple $\sigma_1 = (a, 0, 1, 1, r, 0)$ with $(2a+1, 2r) = 1$ is admissible. Moreover, if $s = -q_{\sigma_1}$, then the 6-tuple $\sigma_n = (a, 0, 1, n, r, s)$ is admissible for each $n > 1$ and the Dunwoody manifold $M_n = M(a, 0, 1, n, r, s)$ is the n -fold cyclic covering of \mathbf{S}^3 , branched over the 2-bridge knot of type $(2a+1, 2r)$. Thus, all branched cyclic coverings of 2-bridge knots are Dunwoody manifolds.*

Proof. From $(2a+1, 2r) = 1$ it immediately follows that σ_1 has a unique cycle in \mathcal{D} . Since $d = 2a+1$ is odd, Corollary 4 proves that σ_1 is admissible. Since $p_{\sigma_n} = p_{\sigma_1} = +1$, all assumptions of Corollary 7 hold; hence σ_n is admissible for each $n > 1$ and M_n is an n -fold cyclic covering of \mathbf{S}^3 , branched over a knot $K \subset \mathbf{S}^3$ which is independent on n . In order to determine this knot, we can restrict our attention to the case $n = 2$. Note that $[s]_2 = [-q_{\sigma_1}]_2 = [b]_2 = 0$ and hence s is always even. Thus, in the case $n = 2$ we can suppose $s = 0$. Let us consider now the genus two Heegaard diagram $H(a, 0, 1, 2, r, 0)$. The sequence of Singer moves [31] on this diagram, drawn in Figures 3–10 and described in the Appendix of the paper, leads to the canonical genus one Heegaard diagram of the lens space $L(2a+1, 2r)$ (see Figure 10). Since the representation of lens spaces (including \mathbf{S}^3) as 2-fold branched coverings of \mathbf{S}^3 is unique [14], the result immediately holds. ■

Remark 5. The Dunwoody manifold $M(a, 0, 1, n, r, s)$ of Theorem 8 is homeomorphic to the Minkus manifold $M_n(2a+1, 2r)$ [21] and the Lins-Mandel manifold $S(n, 2a+1, 2r, 1)$ [19, 24].

An immediate consequence of Theorem 8 is:

Corollary 9 *The fundamental group of every branched cyclic covering of a 2-bridge knot admits a cyclic presentation which is geometric.*

Remark 6. In [21] is shown that the fundamental group of every branched cyclic covering of a 2-bridge knot admits a cyclic presentation, but without pointing out that this presentation is geometric.

About the set \mathcal{K} of knots in \mathbf{S}^3 involved in Corollary 7, we propose the following:

Conjecture. The set \mathcal{K} contains all torus knots.

If this conjecture is true, the set \mathcal{K} contains knots with an arbitrarily high number of bridges. Moreover, the conjecture implies that every branched cyclic covering of a torus knot admits a geometric cyclic presentation. The above conjecture is supported by several cases contained in Table 1 of [6] (see [32]). For example, the Dunwoody manifolds $M(1, 2, 3, n, 4, 4)$ (resp. $M(1, 3, 4, n, 5, 5)$) are the n -fold branched cyclic coverings of the 4-bridge torus knot $K(4, 5)$ (resp. of the 5-bridge torus knot $K(5, 6)$).

4 Appendix

Now we show how to obtain, by means of Singer moves [31] on the genus two Heegaard diagram $H(a, 0, 1, 2, r, 0)$ of Figure 3, the canonical genus one Heegaard diagram of the lens space $L(2a + 1, 2r)$ of Figure 10. The result will be achieved by a sequence of exactly $a + 4$ Singer moves: one of type ID, $a + 2$ of type IC and the final one of type III.

Figure 3 shows the open Heegaard diagram $H(a, 0, 1, 2, r, 0)$. Note that, since $s = 0$, the cycle C'_1 (resp. C'_2) is glued with the cycle C''_1 (resp. C''_2). Let D_1 (resp. D_2) be the cycle of the Heegaard diagram corresponding to the arc e' (resp. e'') coming out from the vertex v' of C'_1 (resp. v'' of C'_2) labelled $a + 1$. Orient D_1 (resp. D_2) so that the arc e' (resp. e'') is oriented from up to down (resp. from down to up). This orientation on D_2 is opposite to the canonical one but, in this way, all the $2a$ arcs connecting C'_1 with C'_2 are oriented from C'_1 to C'_2 and all the $2a$ arcs connecting C''_1 with C''_2 are oriented from C''_2 to C''_1 . The cycle D_1 , besides the arc e' , has two arcs for each $k = 0, \dots, a - 1$, one joining the vertex of C'_1 labelled $a + 1 - (1 + 2k)r$ with the vertex of C'_2 labelled $a + 1 + (1 + 2k)r$, and the other one joining the vertex of C''_2 labelled $a + 1 + (1 + 2k)r$ with the vertex of C''_1 labelled $a + 1 - (3 + 2k)r$. The cycle D_2 , besides the arc e'' , has two arcs for each $k = 0, \dots, a - 1$, one joining the vertex of C'_1 labelled $a + 1 - (2 + 2k)r$ with the vertex of C'_2 labelled $a + 1 + (2 + 2k)r$, the other joining the vertex of C''_2 labelled $a + 1 + 2kr$ with the vertex of C''_1 labelled $a + 1 - (2 + 2k)r$.

The first Singer move consists of replacing the curve D_2 with the curve $D'_2 = D_1 + D_2$ (move of type ID of [31]) obtained by isotopically approaching the arcs e' and e'' until their intersection becomes a small arc and by removing the interior of this arc. The move is completed by shifting, with a small isotopy, D_1 in D'_1 so that it becomes disjoint from D'_2 .

The resulting Heegaard diagram is drawn in Figure 4. The new $2a + 1$

pairs of vertices obtained on C'_1, C''_1, C'_2, C''_2 are labelled by simply adding a prime to the old label, while the $4a + 2$ pairs of fixed vertices keep their old labelling. Note that each new vertex labelled j' is placed, in the cycles C'_1, C''_1, C'_2 and C''_2 , between the old vertices labelled j and $j + 1$ respectively. The cycles C'_2 and C''_2 are no longer connected by any arc, while the cycles C'_1 and C''_1 are connected by a unique arc (belonging to D'_1) joining the vertex labelled $(a + 1)'$ of C'_1 with the vertex labelled $(a + 1 - r)'$ of C''_1 . All the $3a$ arcs connecting C'_1 and C'_2 are oriented from C'_1 to C'_2 and all the $3a$ arcs which now connect C''_1 with C''_2 are oriented from C''_2 to C''_1 . The cycle D'_2 contains exactly $4a + 2$ arcs; more precisely, for each $i = 1, \dots, 2a + 1$, it has one arc joining the vertex labelled i of C'_1 with the vertex labelled $2a + 2 - i$ of C'_2 and one arc joining the vertex labelled i of C''_2 with the vertex labelled $2a + 2 - 2r - i$ of C'_2 . The cycle D'_1 is a copy of the cycle D_1 and hence it contains $2a + 1$ arcs. One of these arcs connects C'_1 with C''_1 ; moreover, for each $k = 0, \dots, a - 1$, D'_1 has one arc joining the vertex of C'_1 labelled $(a + 1 - (1 + 2k)r)'$ with the vertex of C'_2 labelled $(a + 1 + (1 + 2k)r)'$ and one arc joining the vertex of C''_2 labelled $(a + 1 + (1 + 2k)r)'$ with the vertex of C''_1 labelled $(a + 1 - (3 + 2k)r)'$.

Now, apply to the diagram a Singer move of type IC, cutting along the cycle E (drawn in Figure 4) containing C''_1 and C''_2 and gluing the curve C''_2 of the resulting disc with C'_2 .

The new Heegaard diagram obtained in this way shown in Figure 5. It contains the new cycles E' and E'' , which are copies of the cutting cycle E . These cycles replace C'_2 and C''_2 and they both have a unique vertex (w' and w'' respectively). The cycle E' (resp. E'') is connected with C'_1 (resp. with C''_1) by an arc joining w' (resp. w'') with the vertex labelled $(a + 1)'$ (resp. $(a + 1 - r)'$), oriented as in Figure 5. The cycles C'_1 and C''_1 are joined by $3a + 1$ arcs, all oriented from C'_1 to C''_1 ; $2a + 1$ of them belong to D'_2 and the other a belong to D'_1 . More precisely, for each $i = 1, \dots, 2a + 1$, there is an arc of D'_2 joining the vertex labelled i of C'_1 with the vertex labelled $i - 2r$ of C''_1 ; while, for each $k = 0, \dots, a - 1$, there is an arc of D'_1 joining the vertex labelled $(a + 1 - (1 + 2k)r)'$ of C'_1 with the vertex labelled $(a + 1 - (3 + 2k)r)'$ of C''_1 .

Apply again a Singer move of type IC, cutting along the cycle F_1 (drawn in Figure 5) containing C''_1 and E'' and gluing the curve C''_1 of the resulting disc with C'_1 .

The resulting Heegaard diagram is shown in Figure 6. It contains the new cycles F'_1 and F''_1 , which are copies of the cutting cycle F_1 . These cycles

replace C'_1 and C''_1 and they both have one vertex less. It is easy to see that the cycle D'_2 has exactly the same $2a + 1$ arcs connecting F'_1 and F''_1 , all oriented from F'_1 to F''_1 ; if the labelling of the vertices of F'_1 and F''_1 is induced by the labelling of F_1 shown in Figure 5, these arcs join pairs of vertices with the same labelling of the previous step. The cycle D'_1 instead has one arc less than in the previous step. In fact, it has $a - 1$ arcs, connecting F'_1 and F''_1 , all oriented from F'_1 to F''_1 and joining the vertex labelled $(a + 1 - (1 + 2k)r)'$ of F'_1 with the vertex labelled $(a + 1 - (3 + 2k)r)'$ of F''_1 , for $k = 1, \dots, a - 1$.

Now, apply again a Singer move of type IC, cutting along the cycle F_2 (drawn in Figure 6) containing F''_1 and E'' and gluing the curve F''_1 of the resulting disc with F'_1 .

The new Heegaard diagram only differs from the previous one for containing one arc less in the cycle D'_1 . By inductive application of Singer moves of type IC, cutting along the cycle F_h (drawn in Figure 7) containing F''_{h-1} and E'' and gluing the curve F''_{h-1} of the resulting disc with F'_{h-1} , we obtain, for $h = a$, the situation shown in Figure 8, where the cycle D'_1 contains only two arcs, none of which connects F'_a with F''_a .

After the move of type IC corresponding to $h = a + 1$, we obtain the situation of Figure 9 in which the Heegaard diagram contains a pair of complementary handles given by the pair of cycles E', E'' and by the cycle D'_1 , composed by a unique arc connecting E' with E'' . The deletion of this pair of complementary handles (Singer move of type III) leads to the genus one Heegaard diagram drawn in Figure 10, which is the canonical Heegaard diagram of the lens space $L(2a + 1, 2r)$.

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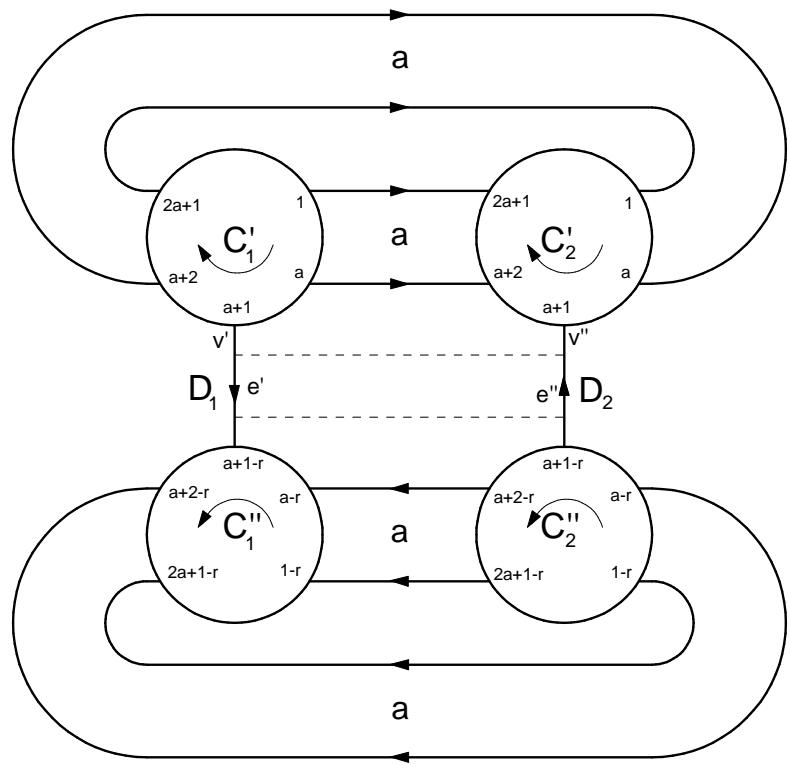


Figure 3:

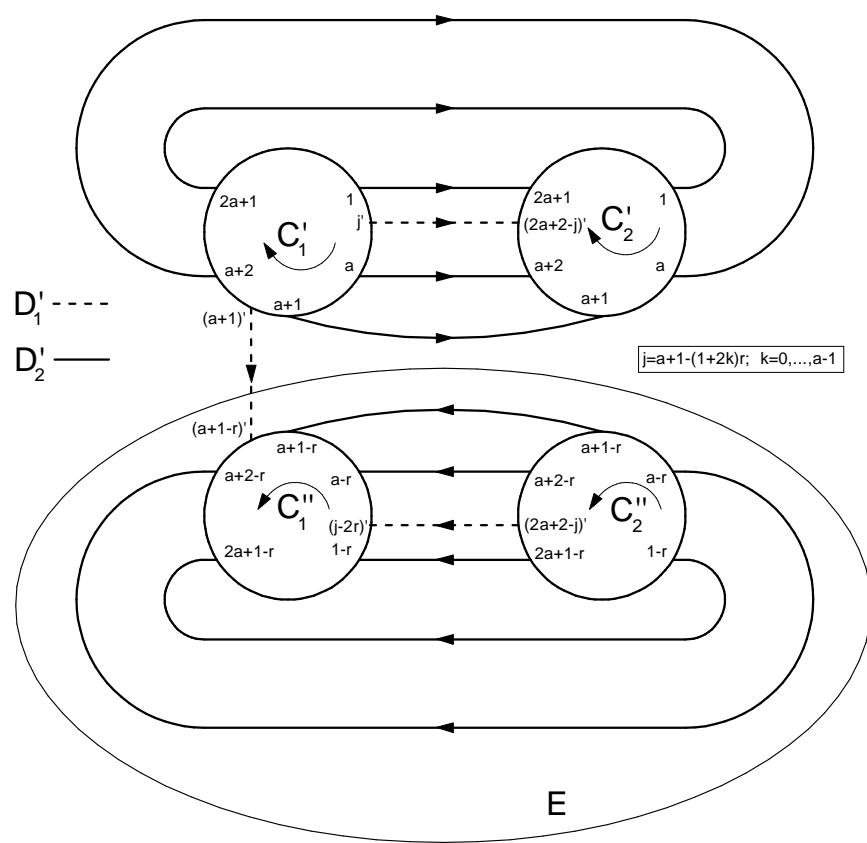


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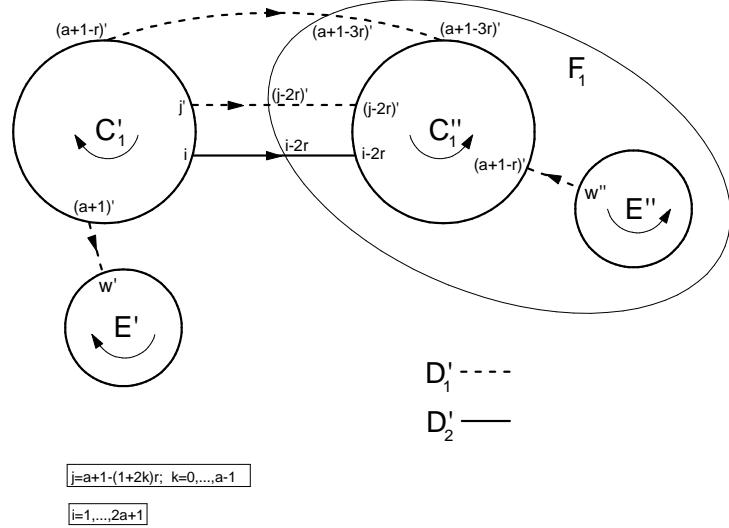


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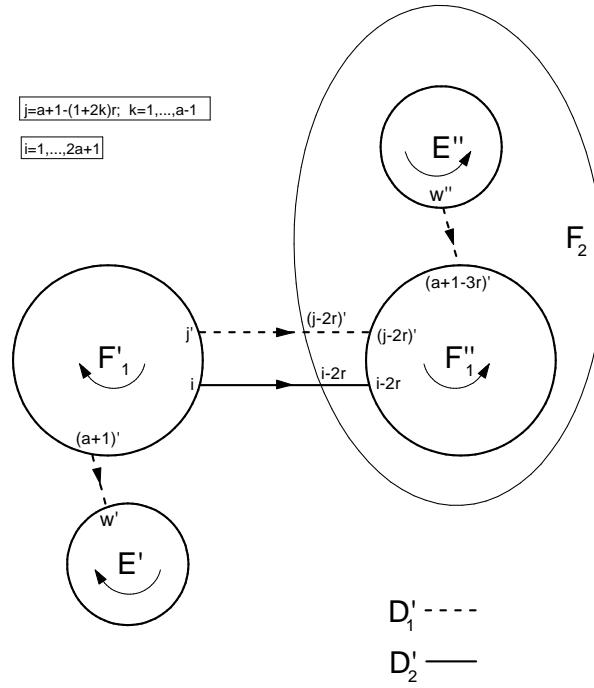


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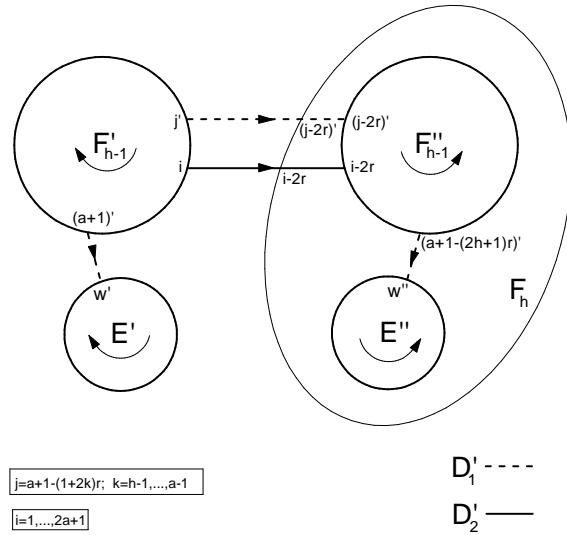


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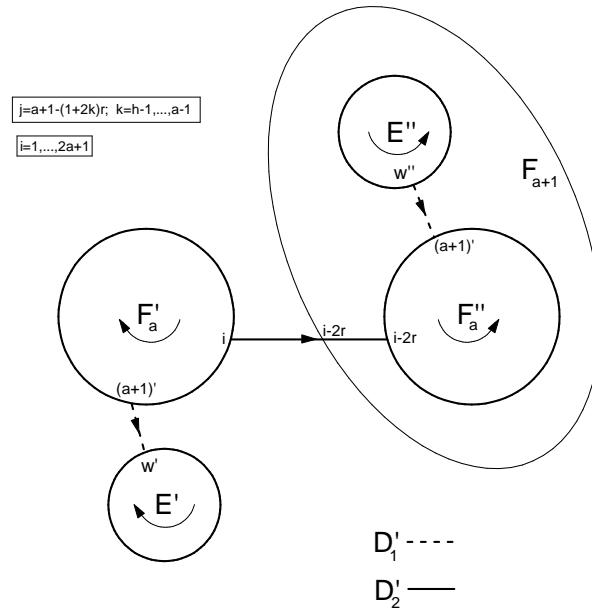


Figure 8:

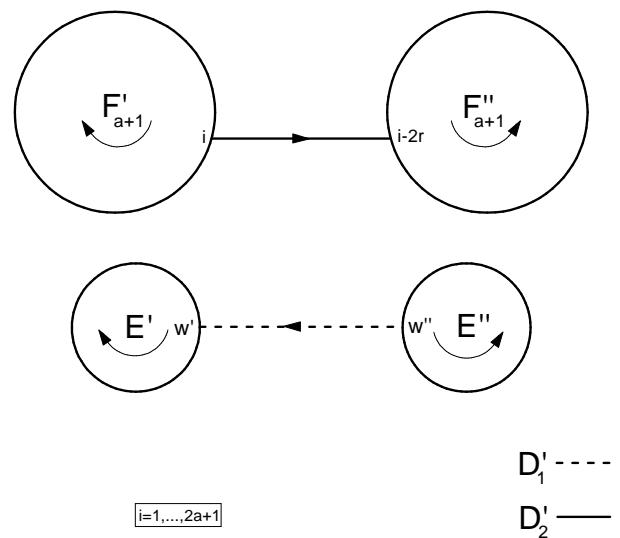


Figure 9:

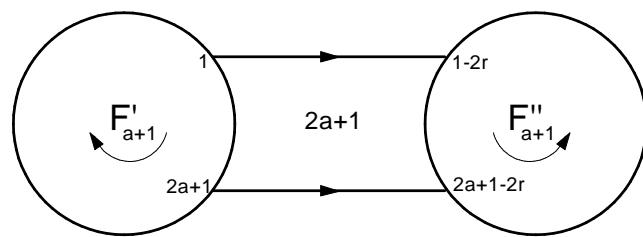


Figure 10: